Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 6: Rules of inference. Section 1.6

1 Rules of inference

Definition 1. An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion. An argument is valid if the truth of all its premises implies that the conclusion is true. An argument form in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Remark 2. From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, \cdots, p_n$ and conclusion q is valid, when $p_1 \wedge p_2 \wedge \cdots \wedge p_n \to q$ is a tautology.

The tautology associated to every rule of inference is:

1. (Modus ponens) $p \wedge (p \to q) \to q$.
2. (Modus tolens) $\neg q \wedge (p \to q) \to \neg p$.
3. (Hypothetical syllogism) $(p \to q) \wedge (q \to r) \to (p \to r)$.
4. (Disjunctive syllogism) $(p \vee q) \wedge \neg p \to q$.
5. (Addition) $p \to (p \vee q)$.
6. (Simplification) $(p \wedge q) \to p$.
7. (Conjunction) $((p) \wedge (q)) \to (p \wedge q)$.
8. (Resolution) $((p \vee q) \wedge (\neg p \vee r)) \to q \vee r$.

Example 3. State which rule of inference is the basis of the following argument: "It is below freezing and raining now. Therefore, it is below freezing now."
Ans: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$$

This argument uses the simplification rule.

Example 4. State which rule of inference is used in the argument: If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.
Ans: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & r \end{array}$$

Hence, this argument is a hypothetical syllogism.

Example 5. Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."
Ans: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q, r \to p, \neg r \to s$ and $s \to t$. The conclusion is simply t. We need to give a valid argument that with our premises gives t:

Step	Rule	Reason
1	$\neg p \wedge q$	Premise
2	$\neg p$	Simplification using (1)
3	$r \to p$	Premise
4	$\neg r$	Modus tollens using (2) and (3)
5	$\neg r \to s$	Premise
6	s	Modus ponens using (4) and (5)
7	$s \to t$	Premise
8	t	Modus ponens using (6) and (7)

Questions:

(1) Show that the premises $(p \wedge q) \vee r$ and $r \to s$ imply the conclusion $p \vee s$.